Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational Physics 220 (2006) 1–5
JOURNAL OF COMPUTATIONAL PHYSICS
www.elsevier.com/locate/jcp

Short note

# A TVD principle and conservative TVD schemes for adaptive Cartesian grids

I.V. Sokolov [a,*], K.G. Powell [a], T.I. Gombosi [a], I.I. Roussev [b]

[a] *Center for Space Environment Modeling, University of Michigan, 2455 Hayward Avenue, Ann Arbor, MI 48109, USA*
[b] *Institute for Astronomy, University of Hawaii, 2680 Woodlawn Drive, Honolulu, HI 96822, USA*

Received 17 March 2006; received in revised form 18 July 2006; accepted 24 July 2006
Available online 1 September 2006

*Keywords:* Total Variation; TVD schemes; Adaptive grids

## 1. Introduction

Modern high-resolution conservative numerical schemes (see [1–3]) are widely used for simulating flows of liquids, gases, or plasmas. They provide a robust and accurate method for capturing discontinuities, such as shock waves. A total-variation-diminishing (TVD) principle is an important element of such schemes. A TVD scheme is constructed in such a way that the total variation, $\mathrm{TV} = \sum_i \|U_i - U_{i-1}\|$, would not increase as the numerical solution is advanced in time. The TVD property of the scheme does not allow the appearance of new local extrema for the grid function $U_i$, eliminating any spurious oscillations, that tend to form, e.g., behind a shock wave.

For uniform Cartesian grids, the TVD property of the high-order scheme can be achieved by employing a Riemann solver together with a limiting procedure for the gradients of the variables. The limiting procedure yields the reconstructed face values that are inputs for the Riemann solver. For example, to limit a slope in the $x$-direction, a limiter function should be applied to two differences: $U_{i,j,k} - U_{i-1,j,k}$ and $U_{i+1,j,k} - U_{i,j,k}$. The resulting algorithms ensure the second order of accuracy in smooth regions of the flow.

The aim of the present paper is to generalize a TVD principle for a special case of piece-wise uniform grids, i.e., adaptive Cartesian grids. Such grids can be thought of as the result of a multiple refinement procedure applied to an originally uniform Cartesian grid. In the course of the refinement procedure, some of the grid cells are split into two equal parts along each (or some) of the spatial directions (see [3–5], for detail). A particular feature of the adaptive Cartesian grid is the interface that occurs at resolution changes, i.e., refinement interface (RI), through which a coarser cell faces its finer neighbors. Without generalizing the TVD concept for such a configuration of the control volumes, the direct application of the same pair limiter functions as for the uniform grid neither provides second-order accuracy nor ensures the stability of the scheme near the RI. We demonstrate how the TVs from the parts of the grid with different levels of refinement can be merged

* Corresponding author. Tel.: +1 734 647 4705.
*E-mail address:* igorsok@umich.edu (I.V. Sokolov).

0021-9991/$ - see front matter 
doi:10.1016/j.jcp.2006.07.021

through the RI and formulate the sufficient conditions for TVD property of the numerical scheme at RIs. The fuller version of the paper is available online [6].

## 2. Total variation for a Cartesian block adaptive grid

Consider a scalar conservation law, for example, a linear advection equation: $\partial U/\mathrm{d}t + a\partial U/\mathrm{d}x = 0$, $a > 0$. Apply a conservative, semi-discrete scheme of the kind: $\mathrm{d}U_i/\mathrm{d}t = S(F_{i-1/2} - F_{i+1/2})/V$, where $V$ and $S$ are the volume and the face area of a cell, respectively. The numerical flux $F_{i-1/2}$, which is the flux through the face between the cells with indexes $i-1$ and $i$, should be obtained by applying a Riemann solver, $F_{\mathrm{RS}}$, to the state variables, and then interpolated to the face position from Left (L) and Right (R) hand sides: $F_{i-1/2} = F_{\mathrm{RS}}(U^{\mathrm{L}}_{i-1/2}, U^{\mathrm{R}}_{i-1/2}) = aU^{\mathrm{L}}_{i-1/2}$. The resulting scheme is as follows:

$$\mathrm{d}hU_i/\mathrm{d}t = aS(U^{\mathrm{L}}_{i-1/2} - U^{\mathrm{L}}_{i+1/2})/V. \tag{1}$$

At RIs, the faces of the coarser cells may consist of subfaces, which are the faces of the finer neighboring cells. Thus, the flux to the finer cell through the RI is $S_f F_{r+1/2}(U^{\mathrm{L}}_{r+1/2,f}, U^{\mathrm{R}}_{r+1/2,f})$. Also, to ensure the total conservation property, the sum of these fluxes over the finer subfaces should be taken as the flux from the coarser cell through the RI, as in [5]. First, let us consider the case of a linear wave propagation from a coarser part of the grid to a finer one. The total variation in this case, TV, is chosen as follows:

$$\mathrm{TV} = \sum_{i=1}^{r} S|U_{i-1} - U_i| + \sum_f S_f\left[|U_r - U_{r+1,f}| + \sum_{i=r+2}^{I+1} |U_{i-1,f} - U_{i,f}|\right]. \tag{2}$$

Here, for simplicity, we consider only a single chain of coarser cells enumerated with an index $i$, $1 \leqslant i \leqslant r$, and having a volume $V$ and a face area $S$ in the transversal cross-section. The last member of the coarser cell chain has a common face with the first members of several finer cell chains at the RI. The finer cells are enumerated with the index $i, r+1 \leqslant i \leqslant I$ in the longitudinal direction and with the (set of) index(es) $f$ in the transversal direction(s) (see Fig. 1). These cells have volumes $V_f$ and face areas $S_f$. Let us work in the 2D case and choose a refinement factor of two. In this case, the finer cells are half the size of the coarser ones, i.e., $V_f = V/4$, $S_f = S/2$, and the index $f$ passes through a set of two values (say, $f = -1/2, 1/2$). With the scheme given by Eq. (1), one can readily evaluate the time derivative of the total variation:

$$\frac{\mathrm{dTV}}{\mathrm{d}t} = \frac{aS^2}{V}\left(\sum_{i=1}^{r-1} T_i + T_r\right) + \sum_f\left[\frac{aS_f^2}{V_f}\left(T_{r+1,f} + \sum_{i=r+2}^{I} T_{i,f}\right)\right], \tag{3}$$

where

$$\begin{aligned}
T_i &= \left(U^{\mathrm{L}}_{i-1/2} - U^{\mathrm{L}}_{i+1/2}\right)[\mathrm{sgn}(U_i - U_{i+1}) - \mathrm{sgn}(U_{i-1} - U_i)],\\
T_r &= \left(U^{\mathrm{L}}_{r-1/2} - \left\langle U^{\mathrm{L}}_{r+1/2,f}\right\rangle\right)\left[\langle \mathrm{sgn}(U_r - U_{r+1,f})\rangle - \mathrm{sgn}(U_{r-1} - U_r)\right],\\
T_{r+1,f} &= \left(U^{\mathrm{L}}_{r+1/2,f} - U^{\mathrm{L}}_{r+3/2,f}\right)\left[\mathrm{sgn}(U_{r+1,f} - U_{r+2,f}) - \mathrm{sgn}(U_r - U_{r+1,f})\right],\\
T_{i,f} &= \left(U^{\mathrm{L}}_{i-1/2,f} - U^{\mathrm{L}}_{i+1/2,f}\right)\left[\mathrm{sgn}(U_{i,f} - U_{i+1,f}) - \mathrm{sgn}(U_{i-1,f} - U_{i,f})\right].
\end{aligned}$$

Herewith, we use the denotation $\langle U^{\mathrm{L}}_{r+1/2,f}\rangle = \sum_f S_f U^{\mathrm{L}}_{r+1/2,f}/S$ for an average over the finer neighbors of the coarser cell weighted by $S_f/S$. The non-positivity of terms like $T_i$, which actually relate to a uniform grid,

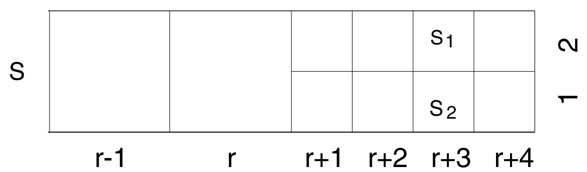


Fig. 1. Schematic picture of the cell enumeration around the RI.

can be ensured using a limited reconstruction procedure for the face values (see [1]). We omit all the derivations for the regions of uniformity. Since the total variation is chosen such that it diminishes with the first-order monotone flux at RI, we also have some freedom for the limited reconstruction of slopes towards the RI. In the first place, $T_r$ can be re-written as follows:

$$T_r = \left(1 + \frac{\langle U^{\rm L}_{r+1/2}\rangle - U_r}{(U_r - U_{r-1})} - \frac{U^{\rm L}_{r-1/2} - U_{r-1}}{U_r - U_{r-1}}\right) N_r,$$

where $N_r = (U_{r-1} - U_r)\langle \mathrm{sgn}(U_r - U_{r+1,f})\rangle - |(U_{r-1} - U_r)| \leqslant 0$. From here, one can find that $T_r$ is negative if at least one of the differences $U_r - U_{r+1,f}$ has a sign different from that of $U_{r-1} - U_r$, and as long as the following conditions are fulfilled:

$$\frac{\langle U^{\rm L}_{r+1/2}\rangle - U_r}{U_r - U_{r-1}} \geqslant 0, \quad \left|\frac{U^{\rm L}_{r-1/2} - U_{r-1}}{U_r - U_{r-1}}\right| < 1. \tag{4}$$

Analogously, $T_{r+1,f} \leqslant 0$ if

$$\frac{U^{\rm L}_{r+3/2,f} - U_{r+1,f}}{U_{r+1,f} - U_r} \geqslant 0, \quad \left|\frac{U^{\rm L}_{r+1/2,f} - U_r}{U_{r+1,f} - U_r}\right| < 1. \tag{5}$$

Now let us discuss the wave propagation from a finer part of the grid to a coarser one. For simplicity, consider the same grid configuration as above, but choose the wave speed to be negative, $a < 0$. It appears that the total variation in this case cannot be used in the same form as for the case $a > 0$. Thus, we consider another variation as follows:

$$\mathrm{TV} = \cdots + \left|SU_r - \sum_f S_f U_{r+1,f}\right| + \cdots. \tag{6}$$

Here we omitted the contributions from the uniform parts of the grid, which are the same as in Eq. (2). In contrast with the case of $a > 0$, here the state in the coarser cell near the RI is coupled with the state that is averaged over the finer neighbors rather than with each of them separately. After evaluating the time derivative $\mathrm{d(TV)}/\mathrm{d}t$ using the upwinded flux $F_{i-1/2} = aU^{\rm R}_{r-1/2}$ for $a < 0$, we find that the defined total variation does not increase, at least in the case when the first order scheme is used with $U^{\rm R}_{i-1/2} = U_i$. The difference between the conservative state in the coarser cell and the one averaged over its finer neighbors should be used in limiting the slopes. Specifically, the higher-order scheme does not increase the total variation if the following limitations for the slopes are fulfilled:

$$\frac{U^{\rm R}_{r-1/2} - U_r}{U_r - \langle U_{r+1,f}\rangle} \geqslant 0, \quad \left|\frac{\langle U^{\rm R}_{r+1/2,f} - U_{r+1,f}\rangle}{U_r - \langle U_{r+1,f}\rangle}\right| < 1, \tag{7}$$

$$\frac{U^{\rm R}_{r+1/2,f} - U_{r+1,f}}{U_{r+1,f} - U_{r+2,f}} \geqslant 0, \quad \left|\frac{U^{\rm R}_{r+3/2,f} - U_{r+2,f}}{U_{r+1,f} - U_{r+2,f}}\right| < 1. \tag{8}$$

## 3. Slope limiters for resolution interfaces

Here the grid configuration near the RIs is not symmetric. Nevertheless, when limiting the slopes we use only symmetric limiter functions $L(\delta U_1, \delta U_2)$, which are such that:

$$L(\delta U_1, \delta U_2) = L(\delta U_2, \delta U_1). \tag{9}$$

The limiter functions used to construct the second-order TVD schemes on a uniform grid have to satisfy the following condition: $L(\delta U_1, \delta U_2)$ should have the same sign as both $\delta U_1$ and $\delta U_2$, or it must equal zero if $\delta U_1$ and $\delta U_2$ are of a different sign, i.e.,

$$L(\delta U_1, \delta U_2) = \frac{\mathrm{sgn}(\delta U_1) + \mathrm{sgn}(\delta U_2)}{2} L(|\delta U_1|, |\delta U_2|). \tag{10}$$

The limiter function should limit the slopes: $M[L] \leqslant 2$, where

$$M[L] = \sup(L(|\delta U_1|, |\delta U_2|)/|\delta U_1|). \tag{11}$$

To ensure the second-order accuracy of the scheme, if the one-side slopes are close to each other then the limiter operator should not distort them, i.e., $L(\delta U_1, \delta U_2) \to \delta U_1$ for $\delta U_2 \to \delta U_1$. A well known example of a limiter function is minmod:

$$\mathrm{minmod}(\delta U_1, \delta U_2) = \frac{\mathrm{sgn}(\delta U_1) + \mathrm{sgn}(\delta U_2)}{2} \min(|\delta U_1|, |\delta U_2|), \tag{12}$$

with $M[\mathrm{minmod}] = 1$. While the minmod is the most robust among all slope limiters, it is the least accurate though. However, at the same time, it satisfies the strongest restriction for $M[L]$. (For the sake of comparison, note that the superbee and analogous limiters have $M[L] = 2$.) Below we consider only Cartesian adaptive grids with an extra restriction: at each RI, the resolution change is not greater than a factor of two. For such grids, we will apply the limiter functions with the following additional restriction:

$$M[L] \leqslant 3/2. \tag{13}$$

Here $3/2 = 1 + \Delta_f/\Delta$, where $\Delta_f/\Delta$ is the size ratio for the neighboring cells. Therefore, limiters like the superbee cannot be applied at the RI. To allow lower size ratios, i.e., $\Delta_f \ll \Delta$, we would have to use even less accurate limiters with $M[L] \approx 1$ (i.e., practically minmod only).

The interpolation procedure in the cells near the RIs is performed in two steps. First, the slopes are limited using the difference $\langle U_{r+1,f}\rangle - U_r$ through RIs. This difference can be used in evaluating slopes along the $x$-direction because the line which connects the center of the coarser cell, $\mathbf{x}_r$, with the mean point for the finer cell centers, $\langle \mathbf{x}_{r+1,f}\rangle$, is parallel to the $x$-axis. This means that:

$$\delta_x U_r = L\left(\frac{1}{2}(U_r - U_{r-1}), \frac{(\Delta_x)_r(\langle U_{r+1,f}\rangle - U_r)}{(\Delta_x)_r + (\Delta_x)_{r+1,f}}\right), \tag{14}$$

and

$$\delta_x U_{r+1,f} = L\left(\frac{(\Delta_x)_{r+1,f}(\langle U_{r+1,f}\rangle - U_r)}{(\Delta_x)_r + (\Delta_x)_{r+1}}, \frac{1}{2}(U_{r+2,f} - U_{r+1,f})\right). \tag{15}$$

Here $(\Delta_x)_r$ is the size of the cell $r$ along the $x$-direction and $\delta_x U_r$ is a limited slope along the $x$-direction, which is such that:

$$U^{\mathrm{R}}_{r-1/2} = U_r - \delta_x U_r. \tag{16}$$

Throughout the uniform parts of the grid, Eq. (16) and the analogous one with the plus sign for $U^{\mathrm{L}}_{r+1/2}$ can be applied to all faces at any direction. However, more care is needed near RIs. In the finer cells, in order to satisfy the TVD conditions given by Eqs. (7) and (8), we limit $\delta_x U_{r+1,f}$ with $U_{r+1,f} - U_r$, so that:

$$U^{\mathrm{R}}_{r+1/2,f} = U_{r+1,f} - \mathrm{minmod}(\delta_x U_{r+1,f}, U_{r+1,f} - U_r), \tag{17}$$

$$U^{\mathrm{L}}_{r+3/2,f} = U_{r+1,f} + \mathrm{minmod}(\delta_x U_{r+1,f}, U_{r+1,f} - U_r). \tag{18}$$

In the coarser cell, the value of $U_r + \delta_x U_r$ interpolated towards the RI gives a good second-order approximation for the center of the face of the coarse cell. To re-interpolate this toward the center of the finer subfaces and to obtain $U^{\mathrm{L}}_{r+1/2,f}$, one can notice that the difference in the radius-vectors from the center of finer subface to the center of the coarse face is the same as that from $\mathbf{x}_{r+1,f}$ to $\langle \mathbf{x}_{r+1,f}\rangle$. From here the second-order approximation gives $U^{\mathrm{L}}_{r+1/2,f} = U_r + \delta_x U_r + U_{r+1,f} - \langle U_{r+1,f}\rangle$. Then, to satisfy the TVD conditions given by Eqs. (4) and (5), we need to limit the transverse gradients:

$$U^{\mathrm{L}}_{r+1/2,f} = U_r + \delta_x U_r + \alpha_f \cdot (U_{r+1,f} - \langle U_{r+1,f}\rangle), \tag{19}$$

where the choice of limiting function $\alpha_f$ is as follows:

$$\alpha_f = \min\left(1, \left|\frac{\delta_x U_r}{U_{r+1,f} - \langle U_{r+1,f}\rangle}\right|\right). \tag{20}$$

Additional limiters in Eqs. (17)–(20) do work only at those RIs for which the slopes from a variable value in a coarser cell to those in the finer neighbors are of a different sign, or differ too much in magnitude. The extra diffusion and lower accuracy of the TVD scheme, in this case, seem to stem from a usual interpolation issue. If the gradients along close directions are quite different, then which of them (if any) can be relied upon? Following the TVD principle allows us at the least to obtain a scheme for this case that is stable.

## 4. Conclusion

With the TVD principle generalized for the case of a Cartesian adaptive grid, we have formulated the limited reconstruction procedure at RIs. This procedure provides stable and accurate numerical solutions when combined with a Godunov-type scheme. The limited reconstruction procedure presented here has been utilized already in the magnetohydrodynamical code BATSRUS at the University of Michigan. Some test results are presented in [6].

## Acknowledgments

We are grateful to P. Colella and G.Toth for fruitful discussions. The development of tools for numerical simulations of the Coronal Mass Ejections in the heliosphere and accompanied particle acceleration processes are funded by NASA-LWS NNG05GM70G and NSF ATM-0454469 (SHINE) grants.

## References

[1] C. Hirsch, Numerical computation of internal and external flowsComputational Methods for Inviscid and Viscous Flows, vol. 2, Wiley, Chichester, 1990, p. 132.

[2] E.F. Toro, Riemann Solvers and Numerical Methods for Fluid Dynamics, Springer, Berlin, 1997.

[3] R.J. LeVeque, Nonlinear conservation laws and finite volume methodsComputational Methods for Astrophysical Fluid Flow, Springer, Berlin, 1998.

[4] U. Trottenberg, C. Osterlee, A. Shueller, Multigrid, Academic Press, San Diego, 2001.

[5] K.G. Powell, P.L. Roe, T.J. Linde, T.I. Gombosi, D.L. De Zeeuw, A solution-adaptive upwind scheme for ideal magnetohydrodynamics, J. Comput. Phys 154 (1999) 284.

[6] I.V. Sokolov, K.G. Powell, T.I. Gombosi, I.I. Roussev, A TVD principle and conservative TVD schemes for adaptive Cartesian grids. Available from: <www.csem.engin.umich.edu/csem/publications/Sokolov2006_TVD.pdf>.